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Decision Aiding

Sequential incorporation of imprecise information in multiple criteria decision processes ☆

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Abstract

In this paper we deal with multicriteria decision processes and develop tools that permit to ease the task of analysing such models. We provide a methodology to sequentially incorporate imprecise preference information which is given by means of general linear relations in the weighting coefficients. The results presented allow us to evaluate the quality of the information supplied and can be used to reduce the number of irrelevant alternatives to be presented to the decision maker (DM). Several examples based on multiple criteria linear programming illustrate the results of the paper.

Keywords: Multicriteria analysis; Imprecise information; Weight restrictions

1. Introduction

Nondominated solutions are the primarily considered solution concept in Multicriteria Decision Problems. However, this approach has as main disadvantage the large number of solutions that it provides to the decision maker (DM). Thus, the study of methods for delimiting the set of solutions to be considered is important in this field, see e.g. Zionts and Wallenius (1976), Steuer (1977), Korhonen and Laasko (1986) and Steuer et al. (1993).

We consider the problem of reducing the set of nondominated solutions in multicriteria decision problems. We investigate the case when the underlying utility function is known to have a linear decomposition, but the weights required to combine the component functions into the real value function are only partially known [see Keeny and Raiffa (1976) for conditions which ensure the linear decomposition].

Although we only consider linear forms for the value functions, many analysts feel that the exact form of the utility function often will not influence the results of the analysis (see e.g. von Winterfeldt and Edwards, 1986; Corner and Buchanan, 1995). Moreover, there are many situations mainly in

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economic and managerial environments, where this hypothesis is quite natural.

We deal with the multicriteria decision problem

$$\mathrm{Max}\,\{f(x),\ x \in X\}$$

where $f : X \to R^k$, and $f_i(x)$, $i = 1, \ldots, k$, is the evaluation of the ith objective for each alternative x of the decision space X.

The nondominated solutions in the weak sense for this problem are given by the preference structure shown in the following relation:

$$x\Re y \iff f(x) - f(y) > 0.$$

For the sake of readability, we will use the term nondominated rather than weakly nondominated.

This preference structure is often insufficient to provide further assistance in choosing among the set of nondominated solutions. Thus, if the DM is able to supply additional information that enriches the preference structure, it must be used in order to reduce the set of alternatives to be considered.

Since we assume the DM's underlying value function is linear, there is a vector

$$\lambda^* \in \Lambda^+ = \left\{ \lambda \in R^k,\ \lambda \geqslant 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

such that the DM preferences are given by the relation $\Re_{\lambda^*}$, as

$$x\Re_{\lambda^*} y \iff \lambda^{*t}(f(x) - f(y)) > 0.$$

Each component of λ^* represents the weight that each criterion brings to the final score of an alternative, and if the functions f are correctly normalised, we can assume that the weights represent the relative importance of the criteria (see Chankong and Haymes, 1983). Hence, if the DM is able to determine these weights, the value function is perfectly defined, and the problem could be analysed as a single objective one.

Needless to say that the exact determination of those weights is difficult and in most cases the DM is only able to give imprecise (incomplete or partial) preference information with respect to its underlying additive utility function. In this sense, we allow the DM to relax the way to supply the weights. In many cases, the only available information consists of bounds for the weights or judgements about the relative importance of the criteria which often can be formalised as linear relations in the weighting coefficients (Steuer, 1976; Mármol et al., 1998).

In general, if we have the information that the weights verify a system of linear inequalities which determine a polyhedron P, constraining the whole set Λ^+, we define the preference between the alternatives by the binary relation $\Re_P$ as

$$x\Re_P y \iff \lambda^{\mathrm{t}}(f(x) - f(y)) > 0 \quad \forall \lambda \in P.$$

As, in general, P has an infinite number of elements, this binary relation is difficult to handle. Nevertheless, if the extreme points, $\lambda^1, \ldots, \lambda^m$, of P are known, the relation $\Re_P$ can be evaluated equivalently as

$$x\Re_P y \iff \lambda^{i^{\mathrm{t}}}(f(x) - f(y)) > 0 \quad \forall i = 1, \ldots, m. \tag{1}$$

This relationship makes possible to manage $\Re_P$ in certain contexts.

The relations that determine the set of weights may be established in a previous step, when the DM supplies information prior to the consideration of the alternatives. In addition, the relations can also be obtained sequentially as information that expresses preferences among alternatives (pairwise comparisons).

Previous papers have dealt with the problem of incorporating information in Multicriteria Decision Processes, see e.g. Bana e Costa (1990), Corner and Buchanan (1995), Hannan (1981), Hazen (1986), Kirkwood and Sarin (1985), Salo and Hämäläinen (1992) and Weber (1985) among others. Recently Athanassopoulos and Podinovski (1997), have discussed multiple criteria models with finite sets of alternatives when information about the weights is assumed to be in the form of arbitrary linear constraints, obtaining conditions for checking dominance and potential optimality of the alternatives. Another method for assessing weights can be seen in Borcherding et al. (1991). In some papers, specially structured information is incorporated in a unique step and the decision process is analysed (Steuer, 1976; Potter and

Anderson, 1980; Arbel, 1989; Carrizosa et al., 1995; Mármol et al., 1998). In these papers it is also discussed on how the DM should be requested to provide this kind of information. Nevertheless, it seems more natural that the DM offers the information in a sequential way. Once he knows the effect on the alternative set of the last information given, he learns more about the decision process and more specialised information can be supplied. This implies that the DM learns and develops the preference information during the modelling process. This process is shown in Fig. 1.

This approach can also be used in other contexts. For instance, in DEA models it provides an easy way to obtain weighted ratings in qualitative multicriteria decision making, generalising the approach shown in Cook and Kress (1992) (see Fernández et al., 2000). Also, in tolerance analysis, when incorporating additional information to obtain more adjusted tolerance bounds, the procedure can be used to deal with the information set (see Mármol and Puerto, 1997).

To illustrate the scheme consider the following four objective Linear Multiobjective Problem taken from Steuer (1976). It is worth noting that in a linear multiobjective problem the preference relation is given by

$$x \Re y \iff C(x-y) > 0,$$

where C is the criterion matrix.

Example 1.

$$\mathrm{Max}\{Cx,\ x \in X\},$$

where the objective matrix C is given by

$$C = \begin{pmatrix} -4 & -2 & 1 & 2 & -4 & -3 & 2 \\ 0 & -3 & -4 & 4 & 5 & -2 & 1 \\ 5 & 5 & 0 & -2 & 3 & 0 & 5 \\ 3 & -3 & 5 & 0 & 2 & 2 & -4 \end{pmatrix}$$

and X is defined by the constraints:

$$\begin{aligned} &7x_1 + 6x_3 + 2x_4 + 5x_5 \leqslant 100, \\ &4x_1 + 7x_6 + 9x_7 \leqslant 100, \\ &5x_1 + 6x_4 \leqslant 100, \\ &9x_3 + 4x_7 \leqslant 100, \\ &2x_1 + 8x_2 + 5x_3 \leqslant 100, \\ &x_1 + 2x_2 + 2x_3 + 6x_4 + 5x_5 + 8x_6 + 5x_7 \leqslant 100, \\ &5x_3 + 3x_5 + 7x_6 \leqslant 100, \\ &x_i \geqslant 0, \quad i = 1, \ldots, 7. \end{aligned}$$

With no additional information, the nondominated set has 52 extreme points. If the analysis of the alternatives obtained is not satisfactory, the DM may be requested to offer additional information in order to find a smaller subset of nondominated alternatives, where he/she is able to choose from.

One way to reduce the scope of the search may be that the DM supplies ordinal information about the criteria, arranging them in increasing order of preference, i.e.

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \lambda_4.$$

The associated polyhedron of weights is

$$P_{L_1} = \{\lambda \in R^4,\ 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \lambda_4,\ e^{t}\lambda = 1\},$$

where $e^{t} = (1,1,1,1)$. The extreme points of P_{L_1} are the columns of matrix L_1

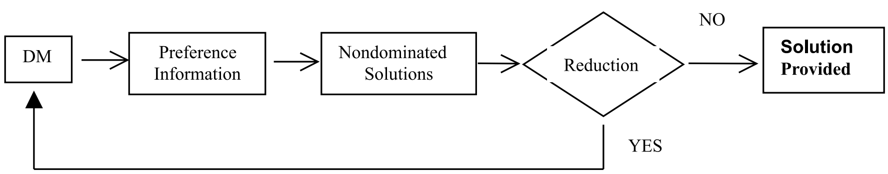

Fig. 1. Sequential incorporation of information.

$$L_1 = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 1/4 & 1/3 & 0 & 0 \\ 1/4 & 1/3 & 1/2 & 0 \\ 1/4 & 1/3 & 1/2 & 1 \end{pmatrix}. \tag{2}$$

Therefore by (1), the nondominated alternatives with respect to the preference structure induced by the ordinal information are the nondominated alternatives of problem $\mathrm{Max}\,\{L_1^t Cx,\ x \in X\}$, that has 14 nondominated extreme points. Thus, the information provided about weights has reduced the set of nondominated alternatives. This process can be repeated as many times as necessary, provided that the DM is able to reduce the set of weights according to the scheme given in Fig. 1.

The goal of this paper is to study the effect of the sequential incorporation of information given by linear relations on the weighting coefficients into the decision process. In the following section we establish the results which are illustrated with examples. The last section is devoted to conclusions. All the proofs are given in Appendix A.

2. Incorporating information to the set of weights

In this section we analyse from a technical point of view the sequential incorporation of information to the process described in Section 1 (see Fig. 1).

Consider a preference structure on Λ^+ given by a system of linear inequalities that reduce the set of weights to the polyhedron $P_L \subset \Lambda^+$. We assume without loss of generality that P_L is given as the convex hull of its extreme points, that are the columns of matrix $L \in R^{k\times p}$. It is worth noting that the knowledge of the extreme points that define a preference structure simplifies its use.

Assume that the DM is willing to improve the preference structure introducing on the current polyhedron of weights, P_L, a new constraint of the form $b \leqslant a^t\lambda \leqslant c$, where $a \in R^k$ is a column vector, and a^t stands for the transpose of a. We are interested in the characterisation of the resulting polyhedron P_H given by

$$P_H = P_L \cap \{\lambda \in R^k,\ \lambda \geqslant 0,\ b \leqslant a^t\lambda \leqslant c\}.$$

2.1. Testing the validity of the information

We classify the linear relations that provide the new information depending on the effect they produce on the actual set of weights. The following result permits us to test the consistency or the redundancy of the new information.

Theorem 1. *Let $v = a^tL$, then*
1. $b \leqslant v_i \leqslant c\ \forall i = 1,\ldots,p \iff P_H = P_L$,
2. $v_i < b\ \forall i = 1,\ldots,p$ *or* $v_i > c\ \forall i = 1,\ldots,p \iff P_H = \emptyset$,
3. *Otherwise* $P_H \neq \emptyset$, *and* $P_H \subset P_L$.

It should be noted that $v = a^tL$ represents the evaluation of the extreme points of P_L on the hyperplane $a^t\lambda = 0$. In case 1, all the extreme points of P_L verify the new constraint, therefore it is redundant. In case 2, the new constraint makes the system inconsistent. Otherwise, the new constraint produces a reduction on the set of weights, and determines a new set of extreme points.

Example 1 (*continued*). Assume that, in addition to the ordinal information, the DM is able to state that the joint importance of the last two objectives is not less than the importance of the first two objectives. Thus, the weights must satisfy

$$\lambda_3 + \lambda_4 \geqslant \lambda_1 + \lambda_2.$$

To analyse how this new constraint changes P_{L_1}, we apply Theorem 1 where $a^t = (-1,-1,1,1)$, $b = 0$, and c is unbounded from above, thus $c = +\infty$.

We calculate $v = a^tL_1 = (0, 1/3, 1, 1)$. It follows, that the new relation is redundant and hence the information polyhedron, P_{L_1}, does not change.

On the contrary, if the DM would have added the information

$$2\lambda_1 \geqslant \lambda_2 + \lambda_3 + \lambda_4,$$

where

$$a^t = (2,-1,-1,-1),\quad b = 0,\quad c = +\infty,$$
$$v = a^tL = (-1/4,-1,-1,-1),$$

the new constraint would have been inconsistent with the previous information given in P_{L_1}. Therefore its incorporation should have been reconsidered.

The possibilities in the case of inconsistency have been previously studied by White et al. (1984), see also Pekelman and Sen (1974). In this sense, the DM is requested to reconsider the information offered with the hope that some of the inconsistent inequalities can be removed or modified in order to produce a nonempty set of weights.

2.2. The new preference structure

Once the results in Theorem 1 ensure an effective reduction of the original preference structure given by P_L, we are interested in the characterisation of the new set of weights P_H. This characterisation will be given obtaining the extreme points of P_H.

Consider the set

$$\Omega_H = \{\omega \in R^p,\ \omega \geqslant 0,\ e^t\omega = 1,\ b \leqslant v\omega \leqslant c\},$$

where $e^t = (1, \dots, 1)$, and notice that by the linear mapping $L : \omega \to \lambda = L\omega$, the set Ω_H is mapped onto the set of weights P_H.

Primarily, we will analyse the relationship between the extreme points of Ω_H and those of P_H. This relationship is very important because we can analyse the preference given by P_H just considering Ω_H, and Ω_H has an easier structure.

Lemma 1. *If $\lambda_0 = L\omega_0$ is an extreme point of P_H, then ω_0 is an extreme point of Ω_H.*

Needless to say that the converse is not always true, as can be seen in the following example.

Example 2. Consider a preference structure given by the polyhedron of weights P_L whose extreme points are the columns of matrix L

$$L = \begin{bmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

Assume that the new additional information is $\lambda_1 \geqslant \lambda_2$. In this case the new set of weights is given by

$$P_H = P_L \cap \{\lambda \in R^3,\ \lambda_1 - \lambda_2 \geqslant 0\}$$

whose extreme points are the columns of the matrix

$$H = \begin{bmatrix} 1 & 1/2 & 1/2 & 1/4 \\ 0 & 1/2 & 0 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix},$$

$$\Omega_H = \{\omega \in R^4,\ \omega \geqslant 0,\ e^t\omega = 1,\ v\omega \geqslant 0\}$$

where

$$v = (1, -1, 0)L = (1, 0, 1/2, -1/2).$$

The extreme points of Ω_H are the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 2/3 & 1/2 \end{pmatrix}.$$

Notice that $(1/3, 0, 0, 2/3)^t$ is an extreme point of Ω_H, but

$$L(1/3, 0, 0, 2/3)^t = (1/3, 1/3, 1/3)^t$$

is not an extreme point of P_H because it is a convex combination of $(1/2, 1/2, 0)^t$ and $(1/4, 1/4, 1/2)^t$.

Although Example 2 shows that, in general, the correspondence between extreme points of Ω_H and P_H is not one-to-one, the following result establishes a condition for the linear transformation L being injective on Λ^+. Injectivity of the application L implies the converse condition of the former Lemma, and it follows that we can obtain the extreme points of P_H from those of Ω_H. This leads us to handle the preference structure given by P_H, only analysing the simpler set Ω_H.

Theorem 2. *If $rank(L) = p$, and ω_0 is an extreme point of Ω_H, then $\lambda_0 = L\omega_0$ is an extreme point of P_H.*

Example 1 (*continued*). Suppose that the DM establishes a new constraint on the set of weights P_{L_1} the first three objectives are jointly more important that the fourth one. The weights must verify

$$\lambda_4 \leqslant \lambda_1 + \lambda_2 + \lambda_3.$$

To incorporate this information to the model we compute

$$a^{\mathrm{t}} = (1,1,1,-1), \quad b = 0, \quad c = +\infty,$$
$$v = a^{\mathrm{t}}L_1 = (1/2, 1/3, 0, -1),$$

where L_1 is matrix (2) defined in Example 1. The extreme points of Ω_H are the columns of matrix

$$\mathrm{ext}(\Omega_H) = \begin{pmatrix} 1 & 0 & 0 & 2/3 & 0 \\ 0 & 1 & 0 & 0 & 3/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/4 \end{pmatrix}.$$

Since $\mathrm{rank}(L_1) = 4$, the condition of Theorem 2 holds. Therefore, the extreme points of the new set of weights are the images of the extreme points of P_H by the linear mapping L_1. The new preference structure is given by the set of weights whose extreme points are the columns of matrix H.

$$H = L_1 \cdot \mathrm{ext}(\Omega_H) = \begin{bmatrix} 1/4 & 0 & 1/6 & 0 \\ 1/4 & 1/3 & 1/6 & 1/4 \\ 1/4 & 1/3 & 1/6 & 1/4 \\ 1/4 & 1/3 & 1/2 & 1/2 \end{bmatrix}.$$

Notice that if the conditions of Theorem 2 hold, the extreme points of P_H are $L_1\omega^s$, where ω^s are the extreme points of Ω_H. But, even if we have not guaranteed the injectivity of L_1 on Λ^+, P_H is the convex hull of $L_1\omega^s$. Nevertheless, some of these points may not be extreme points of P_H. In this case, the use of the complete matrix $L_1\omega^s$ entails the use of a higher number of objectives when solving the associated multicriteria problem. Alternatively, in order to determine which of these points are actually extreme points, one can use the procedure in Edelsbrunner's book (Edelsbrunner, 1987, Theorem 8.11). This procedure has complexity $\mathrm{O}(k\log k + k)$ which is not too large, as in general, the number of objective functions, k, is small.

2.3. Some interesting cases

The previous section has characterised the extreme points of the new preference structure. In this section, we are going to specialise those general results to several cases which appear very often in practice. It is worth noting that we will deal only with Ω_H because the results in Section 2.3 ensure that P_H can be obtained from Ω_H.

The first case characterises the easiest situation for the extreme points of the set Ω_H, i.e. when the extreme point is an element of the canonical basis. Let e^i denote the ith vector of the canonical basis of R^p.

Proposition 1. *e^i is an extreme point of Ω_H if and only if $b \leqslant v_i \leqslant c$, where $v = a^{\mathrm{t}}L$.*

If the new preference information is given by linear inequalities of the form $a^{\mathrm{t}}\lambda \geqslant b$, the following result permits to obtain an explicit expression for the extreme points of Ω_H. Recall that in this case,

$$P_H = \{\lambda \in R^k,\ \lambda = L\omega,\ \omega \geqslant 0,\ e^{\mathrm{t}}\omega = 1,\ b \leqslant v\omega\},$$
$$\Omega_H = \{\omega \in R^p,\ \omega \geqslant 0,\ e^{\mathrm{t}}\omega = 1,\ b \leqslant v\omega\}.$$

Proposition 2. *If $v_i > b$, and $v_j < b$, then*

$$\left(0, \ldots, \underset{(i)}{\frac{b - v_j}{v_i - v_j}}, 0, \ldots, 0, \underset{(j)}{\frac{v_i - b}{v_i - v_j}}, 0, \ldots, 0\right)$$

is an extreme point of Ω_H.

Remark that in this case the whole set of extreme points of Ω_H can be determined. Indeed, we can assume without loss of generality that

$$v = a^{\mathrm{t}}L$$
$$= (v_1, \ldots, v_s, v_{s+1}, \ldots, v_{s+r}, v_{s+r+1}, \ldots, v_{s+r+t}),$$

where

$$v_i = b \quad \forall i = 1, \ldots, s,$$
$$v_i > b \quad \forall i = s+1, \ldots, s+r,$$
$$v_i < b \quad \forall i = s+r+1, \ldots, s+r+t$$

and as a consequence of Propositions 1 and 2, Ω_H has exactly $s + r + (rxt)$ extreme points whose expressions are given by

$$e^i, \quad i = 1, \ldots, s+r$$

$$\left(0, \ldots, \underset{(i)}{\frac{b - v_j}{v_i - v_j}}, 0, \ldots, 0, \underset{(j)}{\frac{v_i - b}{v_i - v_j}}, 0, \ldots, 0\right) \quad \forall i, j,$$

$v_i > b, \; v_j < b.$

Once we have characterised the extreme points of Ω_H, we can get a set of generators of P_H, and if the condition of Theorem 2 holds, we can ensure that they are all extreme points of P_H.

Example 1 (*continued*). Let us assume that the DM wants to add the following relation to the ordinal information

$$\lambda_1 \geqslant 0.5\lambda_2 + 0.5\lambda_3.$$

To analyse how this new constraint changes P_{L_1}, $a^t = (1, -0.5, -0.5, 0)$, $b = 0$, and we calculate $v = a^t L_1 = (0, -1/3, -1/4, 0)$. It follows that the unique extreme points of the associated Ω-set,

$$\Omega_{L_2} = \{\omega \in R^4, \; \omega \geqslant 0, \; e^t\omega = 1, \; v\omega \geqslant 0\},$$

are e^1 and e^4. As the condition stated in Theorem 2 holds, the extreme points of the new set of weights, P_{L_2}, are the columns of matrix L_2:

$$L_2 = L_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 1/4 & 0 \\ 1/4 & 0 \\ 1/4 & 1 \end{pmatrix}.$$

Therefore, in order to solve the 4-objective linear problem proposed in Example 1 with this preference information, we can solve equivalently the 2-objective linear problem $\mathrm{Max}\{L_2^t Cx, \; x \in X\}$, that has 6 extreme efficient points.

Consider now that in a sequential process, the DM incorporates new preference information given by the relation $\lambda_1 + \lambda_2 \geqslant \lambda_4 + 0.1$. Now we get

$$v = (1, 1, 0, -1)^t L_2 = (1/4, -1)$$

The extreme points of the new Ω-set,

$$\Omega_{L_3} = \{\omega \in R^2, \; \omega \geqslant 0, \; e^t\omega = 1, \; v\omega \geqslant 0.1\}$$

are e^1 and the point (1.1/1.25, 0.15/1.25). Therefore the extreme points of the new set of weights are the columns of L_3:

$$L_3 = L_2 \begin{pmatrix} 1 & 1.1/1.25 \\ 0 & 0.15/1.25 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.22 \\ 0.25 & 0.22 \\ 0.25 & 0.22 \\ 0.25 & 0.34 \end{pmatrix}.$$

The resulting MOLP is $\mathrm{Max}\{L_3^t Cx, \; x \in X\}$, that has only two efficient extreme points.

The complete process of sequential incorporation of information to the model, in the case of Example 1 can be synthesised in Table 1.

The extreme points given are represented only by the nonnull coordinates and different extreme points are identified by the superscripts.

The final part of this section is devoted to the particular case where the original preference information P_L is defined by a homogeneous system of linear inequalities, whose matrix has nonnegative inverse. This kind of relations has been already used in the literature of MCDM (Carrizosa et al., 1995; Mármol et al., 1998), and it includes as a particular instance the case of ordinal information (see Cook and Kress, 1992; Kirkwood and Sarin, 1985; Paelinck, 1975, among others).

We consider a preference information structure P_L given by a system of linear inequalities $M\lambda \geqslant 0$, where $M^{-1} \geqslant 0$. In this case, the set of extreme points of P_L are the columns of M^{-1} normalised to add one (Carrizosa et al., 1995).

Let $a^t\lambda \geqslant 0$ be the new constraint incorporated to the set of weights. According to our notation, the new preference structure P_H is

Table 1
Nondominated set

No information	52 extreme points		
Ordinal information:			
$\lambda_4 \geqslant \lambda_3 \geqslant \lambda_2 \geqslant \lambda_1 \geqslant 0$	14 extreme points		
$\lambda_4 + \lambda_3 \geqslant \lambda_2 + \lambda_1$	Redundancy		
$2\lambda_1 \geqslant \lambda_2 + \lambda_3 + \lambda_4$	Inconsistency		
$2\lambda_1 \geqslant \lambda_2 + \lambda_3$	6 extreme points		
	$x_2^1 = 6.76$		$x_3^3 = 8.16$
	$x_5^1 = 10.54$	$x_5^2 = 20$	$x_5^3 = 10.2$
	$x_7^1 = 8.1$		$x_6^3 = 4.08$
	$x_3^4 = 11.11$	$x_3^5 = 11.11$	$x_1^6 = 4.76$
	$x_4^4 = 3.07$	$x_5^5 = 6.67$	$x_3^6 = 11.11$
	$x_5^4 = 5.44$	$x_6^5 = 3.49$	$x_6^6 = 6.35$
	$x_6^4 = 4.02$		
$\lambda_1 + \lambda_2 \geqslant \lambda_4 + 0.1$	2 extreme points		
	$x_2^1 = 6.76$		
	$x_5^1 = 10.54$	$x_5^2 = 20$	
	$x_7^1 = 8.1$		

$$P_H = P_L \cap \{\lambda \in R^k : a^t\lambda \geqslant 0\}.$$

Then, using the tools developed in this section, we can obtain explicitly the set of extreme points of P_H by

$$Le^i \quad \forall i, v_i \geqslant 0.$$

$$L\left(0, \ldots, \underset{(i)}{\frac{-v_j}{v_i - v_j}}, 0, \ldots, 0, \underset{(j)}{\frac{v_i}{v_i - v_j}}, 0, \ldots, 0\right) \quad \forall i, j,$$

$$v_i > 0, \quad v_j < 0.$$

It should be noted that the results established in Mármol et al. (1998) are particular cases of this situation.

3. Conclusions

Many of the real world decision processes involve more than one criterion. For this reason, the development of tools that ease the difficulties that arise when handling several criteria is an important task in the Operations Research field.

When preference information is available, it must be used to reduce the set of nondominated alternatives to be considered. To this end, we deal with the process of incorporation available information in MCDM. We analyse this process from a sequential point of view because it seems more natural for the DM to offer information once he/she knows the effect produced by his/her last interaction with the model. The process should start from a simple preference structure. In particular, it may start from some ordinal relationships among the criteria, and sequentially improve the preference incorporating new linear relations.

In this paper we address the improvement of preference information when it is provided in the form of general linear relations between the weighting coefficients. The results that we present allow us to test the quality of the information provided, as well as they allow us to incorporate the information to the model when it reduces the nondominated set.

The incorporation of new information does not augment the difficulty of the problem that must be solved, and in many cases it leads to simpler models.

The methodology proposed can be used in any interactive procedure in order to improve the quality of the nondominated alternative set in multicriteria decision problems. It also can be applied to obtain the set of feasible decisions corresponding to prior information specified as a set of prior probabilities (see e.g. Potter and Anderson, 1980).

Appendix A

Proof of Theorem 1. Since $v = a^tL$, then

$$P_L = \{\lambda \in R^k,\ \lambda = L\omega,\ \omega \in \Lambda^+\},$$
$$P_H = \{\lambda \in R^k,\ \lambda = L\omega,\ \omega \geqslant 0,\ e^t\omega = 1,\ b \leqslant v\omega \leqslant c\}.$$

(1) If $b \leqslant v_i \leqslant c\ \ \forall i = 1, \ldots, p$ then $b \leqslant vw \leqslant c$ $\forall \omega \in \Lambda^+$, what means $P_L \subset P_H$.

Conversely, since $P_L = P_H$ then $b \leqslant v\omega \leqslant c$ $\forall \omega \in \Lambda^+$, in particular, it holds for $\omega = e^i$, where e^i is the ith vector of the canonical basis of R^p, therefore $b \leqslant v_i \leqslant c\ \forall i = 1, \ldots, p$.

(2) If $v_i < b\ \forall i = 1, \ldots, p$ or $v_i > c\ \forall i = 1, \ldots, p$, then $v\omega < b$ or $v\omega > c$ for all $\omega \in \Lambda^+$ therefore, the set

$$\Omega_H = \{\omega \in R^p,\ \omega \geqslant 0,\ e^t\omega = 1,\ b \leqslant v\omega \leqslant c\}$$

is empty, and it follows that $P_H = \emptyset$.

Conversely, if $P_H = \emptyset$, then $\Omega_H = \emptyset$. Let us suppose that $v_i < b$ and $v_j > c$, and consider

$$\bar{\omega} = \left(0, \ldots, \underset{(i)}{\frac{b - v_j}{v_i - v_j}}, 0, \ldots, 0, \underset{(j)}{\frac{v_i - b}{v_i - v_j}}, 0, \ldots, 0\right).$$

The vector $\bar{\omega}$ verifies $\bar{\omega} \geqslant 0$, $e^t\bar{\omega} = 1$, $v\bar{\omega} = b$, thus $\bar{\omega} \in \Omega_H$ what contradicts the stated hypothesis, and the result follows. □

Proof of Lemma 1. Let us suppose that ω_0 is not an extreme point of Ω_H, then for some $\omega^1, \omega^2 \in \Omega_H$, $\omega_0 = \gamma\omega^1 + (1-\gamma)\omega^2$, $\gamma \in [0, 1]$. Since

$$\begin{aligned}\lambda_0 = L\omega_0 &= L(\gamma\omega^1 + (1-\gamma)\omega^2)\\ &= \gamma L\omega^1 + (1-\gamma)L\omega^2 = \gamma\lambda^1 + (1-\gamma)\lambda^2\end{aligned}$$

with $\lambda^1, \lambda^2 \in P_H$, then λ_0 is not an extreme point of P_H. □

The next result is a technical lemma which gives a necessary and sufficient condition on injectivity of linear mappings in sets of weights.

Let $A \in R_+^{k\times p}$ be a matrix whose columns satisfy $\sum_{i=1}^k a_{ij} = 1\ \forall j = 1, \ldots, p$, and denote by rank($A$) the rank of matrix A.

Lemma 2. *The linear transformation given by the matrix A is injective on the set Λ^+ if and only if rank$(A) = p$.*

Proof of Lemma 2. First of all we will prove that rank$(A) = p$ is a sufficient condition for the linear transformation $A :\rightarrow \lambda = L\omega$ being injective on the set Λ^+. Indeed, consider $\omega^1, \omega^2 \in \Lambda^+$, such that $A\omega^1 = A\omega^2$ (or equivalently $A(\omega^1 - \omega^2) = 0$). Since we assume that rank$(A) = p$, the linear system $A\omega = 0$ has a unique solution $\omega = 0$. Hence $\omega^1 - \omega^2 = 0$, i.e. $\omega^1 = \omega^2$ what means that A is injective on Λ^+.

Conversely, let us suppose that rank$(A) < p$. Consider the matrix

$$A_1 = \begin{pmatrix} A \\ 1, \ldots, 1 \end{pmatrix},$$

and notice that rank(A_1) = rank$(A) < p$. Therefore there exists $z \in R^p$, $z \neq 0$, such that $A_1z = 0$, that is $Az = 0$ and $e^tz = 0$. Now let u^1, u^2 be two vectors such that

$$u_i^1 = z_i \text{ when } z_i > 0 \text{ and } u_i^1 = 0 \text{ when } z_i \leqslant 0,$$
$$u_i^2 = -z_i \text{ when } z_i < 0 \text{ and } u_i^2 = 0 \text{ when } z_i \geqslant 0,$$

then $z = u^1 - u^2$, $e^tu^1 = e^tu^2 = \alpha$. Let $\omega^1 = u^1/\alpha$, $\omega^2 = u^2/\alpha$, then $\omega^1, \omega^2 \in \Lambda^+$ and $\omega^1 \neq \omega^2$, but $A(\omega^1 - \omega^2) = Az/\alpha = 0$, what implies that $A\omega^1 = A\omega^2$, and A is not injective on Λ^+. □

Proof of Theorem 2. Consider that $\lambda_0 = L\omega_0$ is not an extreme point of P_H. We will prove that if $\text{rank}(L) = p, \omega_0$ is not an extreme point of Ω_H.

Let us suppose that $\lambda_0 = \gamma\lambda^1 + (1-\gamma)\lambda^2$, $\gamma \in [0,1]$ for some $\lambda^1, \lambda^2 \in P_H$. In this case, there exist $\omega^1, \omega^2 \in \Omega_H$ such that $\lambda^1 = L\omega^1, \lambda^2 = L\omega^2$. Hence

$$\begin{aligned} L\omega_0 = \lambda_0 &= \gamma L\omega^1 + (1-\gamma)L\omega^2 \\ &= L(\gamma\omega^1 + (1-\gamma)\omega^2) \end{aligned}$$

and, as L is injective on Ω_H, by Lemma 1, we have $\omega_0 = \gamma\omega^1 + (1-\gamma)\omega^2$. Therefore ω_0 is not an extreme point of Ω_H and the result follows. □

Proof of Proposition 1. Let $L^i = Le^i$. Then $v_i = a^tLe^i = a^tL^i$. If we assume that $b \leqslant v_i \leqslant c$ then, as L^i is an extreme point of P_H, from Theorem 1, e^i is an extreme point of Ω_H. Conversely, if e^i is in Ω_H then $b \leqslant ve^i \leqslant c$, and the result follows. □

Proof of Proposition 2. The segment joining e^i and e^j is an extreme edge of the polyhedron Λ^+. It is straightforward to see that the intersection of Λ^+ with hyperplane $v\omega = b$ is given by the point

$$\left(0, \ldots, \underset{(i)}{\frac{b - v_j}{v_i - v_j}}, 0, \ldots, 0, \underset{(j)}{\frac{v_i - b}{v_i - v_j}}, 0, \ldots, 0\right)$$

and it is an extreme point of Ω_H because e^j does not belong to Ω_H. □

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